Math 6082 2025-02-28

§4.3 Computation of the Kolmogrov-Sinai's entropy.

- Recall that for a MPS $(X, \mathcal{B}, \mu, T)$, the entropy of T w.r.t μ is defined by

$$h_\mu(T) = \sup h_\mu(T, \mathcal{D}),$$

where $\mathcal{D}$ ranges over all finite partitions of X, and

$$h_\mu(T, \mathcal{D}) := \lim_{n\to\infty} H_\mu(\mathcal{D} \vee T^{-1}\mathcal{D} \vee \cdots \vee T^{-(n-1)}\mathcal{D}).$$

Below we give a useful proposition.

Prop. 4.6. Let $T: X \to X$ be a cts map on a compact metric space and that $\mathcal{D}_n$ is a sequence of partitions with $\operatorname{diam} \mathcal{D}_n \to 0$. Then

$$h_\mu(T) = \lim_{n\to\infty} h_\mu(T, \mathcal{D}_n).$$

The proof of the prop. is based on several lemmas.

Lem 4.7. Let X be a compact metric space, and let μ be a Borel prob. measure on X. Let

$$G = \{C_1, \cdots, C_n\}$$

be a Borel partition of X. Suppose that $\{\mathcal{D}_m\}_{m=1}^{\infty}$ be a sequence of partitions such that

$$\operatorname{diam} \mathcal{D}_m := \max_{D \in \mathcal{D}_m} \operatorname{diam} D \to 0 \quad \text{as } m \to \infty.$$

Then $\exists$ partitions $\{E_1^m, \cdots, E_n^m\}$ so that

1. each E_i^m is a union of members of $\mathcal{D}_m$
2. $\lim_{m \to \infty} \mu(E_i^m \triangle C_i) = 0$ for each i.

Pf. Pick compact $K_1, K_2, \cdots, K_n$ with $K_i \subset C_i$ and

$$\mu(C_i \setminus K_i) < \varepsilon.$$

Let $\delta = \inf_{i \neq j} d(K_i, K_j)$. Consider m such that $\operatorname{diam} \mathcal{D}_m < \delta/2$.

Divide the elements $D \in \mathcal{D}_m$ into groups whose unions are $E_1^m, E_2^m, \cdots, E_n^m$ so that $D \in E_i^m$ if

$D \cap K_i \neq \emptyset$. As $\operatorname{diam} \mathcal{D}_m < \delta/2$, any $D \in \mathcal{D}_m$ intersects at most one K_i. put a D hitting no K_i in any E_i^m as you like. Then $E_i^m \supseteq K_i$, and

$$\mu(E_i^m \Delta C_i) = \mu(C_i \setminus E_i^m) + \mu(E_i^m \setminus C_i)$$

$$\leq \mu(C_i \setminus K_i) + \mu(X \setminus \bigcup_{i=1}^n K_i)$$

(notice that $E_i^m \setminus C_i \subset X \setminus \bigcup_{i=1}^n K_i$)

$$\leq \varepsilon + n\varepsilon$$

$$= (n+1)\varepsilon.$$

$\blacksquare$

Lem 4.8 Let X be a compact metric space and $\mu \in \mathcal{P}(X)$. Let $\varepsilon > 0$ and $\mathcal{C}$ be a finite Borel partition. There is $\delta > 0$ such that $H_\mu(\mathcal{C} \mid \mathcal{D}) < \varepsilon$ whener $\mathcal{D}$ is a partition with $\operatorname{diam}(\mathcal{D}) < \delta$.

Pf. Let $\mathcal{C} = \{C_1, \cdots, C_n\}$. In Lem 4.7, we have shown that for any $\alpha > 0$, one could find $\delta > 0$ such that whenever $\mathscr{D}$ satisfies $\operatorname{diam}(\mathscr{D}) < \delta$, then there is

$$\mathcal{E} = \{ E_1, \cdots, E_n \} \subset \mathscr{D}$$

(i.e. each E_i is a Union of members in $\mathscr{D}$).

with

$$\mu(E_i \triangle C_i) < \alpha.$$

Now

$$H_\mu(G \mid \mathscr{D}) \leq H_\mu(G \mid \mathcal{E})$$

$$= \sum_i \sum_j \mu(E_i)\, \phi\left(\frac{\mu(C_j \cap E_i)}{\mu(E_i)}\right)$$

Clearly the expression for $H_\mu(G \mid \mathcal{E})$ is a cts function

on $\mu(E_i)$ and $\mu(C_j \cap E_i)$

and vanishes if $\mu(C_j \cap E_i) = \delta_{i,j}\, \mu(E_i)$.

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence when α is small,

$$H_\mu(G \mid \mathcal{E}) < \varepsilon,$$

So $H_\mu(G \mid \mathscr{D}) < \varepsilon$. $\blacksquare$

Pf of prop 4.6:

clearly $h_\mu(T) \geq \limsup_{n\to\infty} h_\mu(T, \mathcal{D}_n)$.

For any partition $\mathcal{G}$,

$$h_\mu(T, \mathcal{G}) \leq h_\mu(T, \mathcal{D}_n) + H_\mu(\mathcal{G} | \mathcal{D}_n) \quad \text{(by Lem 4.5)}$$

By Lem 4.8,

$$h_\mu(T, \mathcal{G}) \leq \liminf_{n\to\infty} h_\mu(T, \mathcal{D}_n).$$

Hence

$$h_\mu(T) = \sup_{\mathcal{G}} h_\mu(T, \mathcal{G}) \leq \liminf_{n\to\infty} h_\mu(T, \mathcal{D}_n).$$

$\blacksquare$

Def. A mapping $T: X \to X$ is called expansive if $\exists\ \varepsilon > 0$ such that

$$d(T^k x, T^k y) \leq \varepsilon \text{ for all } k \geq 0 \Rightarrow x = y.$$

Prop. 4.9 Suppose $T: X \to X$ is a cts transformation on a compact metric space with expansive constant ε. Then $h_\mu(T) = h_\mu(T, \mathcal{D})$ whenever $\operatorname{diam} \mathcal{D} < \varepsilon$.

Pf. Let $\mathcal{D}_n := \mathcal{D} \vee T^{-1}\mathcal{D} \vee \cdots \vee T^{-(n-1)}\mathcal{D}$.

We claim $\operatorname{diam}(\mathcal{D}_n) \to 0$.

To see this, suppose on the contrary that $\operatorname{diam}(\mathcal{D}_n) \not\to 0$.

Then $\exists\ (n_k) \uparrow \infty$, and $x_{n_k}, y_{n_k} \in X$ s.t. $d(x_{n_k}, y_{n_k}) > \delta$,

$$d(T^i x_{n_k}, T^i y_{n_k}) \leq \varepsilon \quad \text{for } 0 \leq i \leq n_k - 1$$

WLOG, assume $x_{n_k} \to x$, $y_{n_k} \to y$.

Then $d(x, y) > \varepsilon$,

$d(T^i x, T^i y) \leq \varepsilon$, $\forall\ i \geq 0 \Rightarrow x = y$ by expansiveness.

Leading to a contradiction.

Since $\operatorname{diam} \mathcal{D}_n \to 0$,

$$h_\mu(T) = \lim_{n\to\infty} h_\mu(T, \mathcal{D}_n).$$

However, $h_\mu(T, \mathcal{D}_n) = h_\mu(T, \mathcal{D})$ by Lem 4.5.

Hence
$$h_\mu(T) = h_\mu(T, \mathcal{D}).$$

§ 4.4. Examples.

Consider the left shift $\sigma : \Sigma \to \Sigma$, where $\Sigma = \{1, \cdots, k\}^{\mathbb{N}}$ is the one-sided full shift space.

Then σ is expansive with constant $1/2$.

- The one-sided $(p_1, \cdots, p_k)$-shift over $\Sigma = \{1, \cdots, k\}^{\mathbb{N}}$ has entropy $-\sum_{i=1}^{k} p_i \log p_i$.

- The one-sided $(\vec{p}, P)$ Markov shift has entropy

$$-\sum_{i,j} p_i P_{ij} \log P_{i,j}.$$